Int. J. Heat Mass Transfer. Vol. 15, pp. 2663–2667. Pergamon Press 1972. Printed in Great Britain

SHORTER COMMUNICATIONS

APPLICATION OF OLFE'S MODIFIED DIFFERENTIAL APPROXIMATION TO THE RADIATION-LAYER PROBLEM ON A FLAT PLATE

WERNER KOCH

DFVLR-Institut für Theoretische Gasdynamik Theaterstrasse 13, D 51-Aachen, Federal Republic of Germany

(*Received* 17 *April* 1972)

NOMENCLATURE

A,	$\frac{3}{2}\, a/(2 - a)$;
a,	total hemispherical absorptivity for a gray surface;
Bo,	$\dfrac{\rho^*_\infty c^*_{p\infty} U^*_\infty}{\sigma^* T^{*3}_\infty}$ Boltzmann number;
Bu,	$\alpha^*_\infty L^*$ Bouguer number;
C_1, C_2,	integration constants;
c^*_p,	specific heat at constant pressure;
$E_n(z)$,	exponential integral of order n;
I^*,	frequency-integrated specific intensity;
$I_n(z)$,	modified Bessel function of order n;
L^*,	plate length;
$\vec{l}^*$,	unit vector in s^*-direction;
m,	$Bu\{3p/(p + 16Bu/Bo)\}^{\frac{1}{2}}$;
p,	Laplace transformation variable;
q',	$q^*_y/\{4\sigma^* T^{*3}_\infty (T^*_P - T^*_\infty)\}$ nondimensional radiative heat flux normal to plate;
q'_g, q'_{ext},	heat flux due to gas and wall radiation respectively;
$\vec{s}^*$,	propagation direction of I^*;
T^*,	absolute temperature;
t,	dummy variable of integration;
U^*,	(constant) fluid velocity parallel to plate;
$x = x^*/L^*$,	$y = y^*/L^*$, dimensionless coordinates parallel and perpendicular to plate respectively;
α^*,	volumetric absorption coefficient for a gray medium;
θ',	$(T^* - T^*_\infty)/(T^*_P - T^*_\infty)$ dimensionless temperature;
ξ,	$\dfrac{2\sigma^*\alpha^*_\infty T^{*3}_\infty x^*}{\rho^*_\infty c^*_{p\infty} U^*_\infty} \equiv 2\, Bux/Bo$ dimensionless variable;
$d\Omega$,	element of solid angle;
ρ^*,	gas density;
σ^*,	Stefan–Boltzmann constant.

Superscripts

$*$,	dimensional quantity;
$\rightarrow$,	vector quantity;
$'$,	perturbation quantity;
$-$,	Laplace transformed quantity.

Subscripts

P,	value on the plate;
∞,	free stream value.

INTRODUCTION

As a result of applications in reentry physics, increased attention has been paid to combined convection and radiation heat transfer. In particular, several recent survey articles and radiation heat transfer books ([1–3], to only list a few) deal with the interaction of radiation and convection in the laminar boundary layer on a flat plate. The complexity of this problem is such that, to date, analytic results can only be obtained if a large number of often unrealistic simplifications is made. One of these basic assumptions, used in almost all investigations but difficult to justify on physical grounds in many situations, is the one dimensionality of the radiation field.

The treatment of the two-dimensional radiation field in the above mentioned problem is manageable only by using the differential approximation (see for example [4, 5]), but, as noted by various authors, large errors occur at intermediate and small optical depths around the tip of the plate. This failure of the differential approximation is due to the strongly non-isotropic nature of the intensity a few photon mean free paths within the leading edge of the isothermal plate, so that the assumptions in the derivation of the differential approximation are violated. Among the several proposed schemes for improving the differential approximation one of the first was a modification due to Olfe [6]†, and although it also has it's limitations it appears to be relevant for our hot-plate problem. So, before trying to solve the two-dimensional problem it would be of interest to quantitatively assess the accuracy of the modified differential approach in this specific case. Such an assessment is possible for the linearized, one-dimensional approximation, namely the radiation-layer problem formulated by Cess [1] because an exact solution for that problem has been published by Williams [8] in the meantime. Furthermore, a numerical solution for the non-linear radiation-layer problem has been given by Taitel [9].

FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

Using Cess' radiation-layer concept, which divides the temperature field into a conventional thermal boundary layer and a radiation layer, we are interested only in finding the solution for the latter. Thus, we consider the flow of a nonviscous, nonconducting and nonscattering, emitting–absorbing, gray medium past a blackbody, isothermal plate of length L^* and at zero incidence. Since the velocity is uniform everywhere the governing energy equation can be written

$$\rho^* c_p^* U_\infty^* \frac{\partial T^*}{\partial x^*} + \operatorname{div} \vec{q}^* = 0. \tag{1}$$

This equation is coupled with the radiation field through the radiative heat flux vector $\vec{q}^*$, and in order to complete our formulation we have to add the definition

$$\vec{q}^* = \int_0^{4\pi} \vec{l}\, I^*(s^*, \vec{l}^*)\, d\Omega \tag{2}$$

and the equation of radiative transfer, which for a gray gas in local thermodynamic equilibrium is

$$\frac{dI^*}{ds^*} = \alpha^* \left\{ \frac{\sigma^*}{\pi} T^{*4} - I^* \right\}. \tag{3}$$

$\vec{l}^*$ is the unit vector in the propagation direction of the frequency-integrated specific intensity I^* and s^* denotes the distance along that ray.

For problems where I^* changes only slowly with direction, an appropriately averaged intensity can be introduced, thus leading to the differential approximation ([4, 5]). Under the assumption of only small deviations from radiative equilibrium, i.e. $(T_P^* - T_\infty^*)/T_\infty^* \ll 1$, we linearize our equations. Then, neglecting the radiative transfer in x-direction we obtain for (1) in terms of our non-dimensional perturbation quantities defined in the Nomenclature

$$Bo \frac{\partial \theta'}{\partial x} + 4 \frac{dq'}{dy} = 0. \tag{4}$$

Similarly, after eliminating the average intensity, the differential approximation results in

$$\frac{d^2 q'}{dy^2} = 4Bu \frac{\partial \theta'}{\partial y} + 3Bu^2 q'. \tag{5}$$

Bo denotes the Boltzmann number and Bu the Bouguer number based on the plate length L^*. As in ordinary boundary layer theory the L^*-dependence of the results cancels for one-dimensional radiation and Bo and Bu can be combined to form the "radiation-layer variable" ξ introduced by Cess [1].

The strongly non-isotropic radiation from the plate invalidates the usual differential approximation a few photon mean free paths within the leading edge. We overcome this difficulty by applying Olfe's modification [6] which consists of dividing the net radiative flux at each point into a contribution q_g' from the almost isotropically radiating medium and the contribution q_{ext}' from the anisotropic wall radiation. Writing

$$q' = q_g' + q_{ext}' \tag{6}$$

in the energy equation (4), q_g' is governed by the differential approximation (5) while q_{ext}' must be computed from (2) with I^* being the solution of (3) for zero gas emission (i.e. the emission term $\sigma^* T^{*4}/\pi$ is omitted). In our case, assuming one-dimensional dependence and, for simplicity, a blackbody plate we find (see [6])

$$q_{ext}' = 2E_3(Bu \cdot y), \tag{7}$$

where $E_n(z)$ is the exponential integral of order n defined, for example, in [10].

Finally, we prescribe the pertinent boundary conditions: As $y \to \infty$ all perturbation quantities have to vanish. For the boundary condition on the plate we write within the differential approximation (see for example [5])

$$q_P' = \frac{4}{3} A \left\{ 1 - \theta_P' + \frac{1}{4Bu} \left(\frac{dq'}{dy} \right)_P \right\}. \tag{8}$$

The constant A is defined in the Nomenclature. When computing q_g' according to the modified differential approximation we omit the wall radiation term $4A/3$ on the right-hand side of (8) since q_{ext}' takes care of that. Furthermore, neglecting the radiation transfer in x-direction we state the "initial condition"

$$\theta'(0, y) = 0. \tag{9}$$

With q_{ext}' given by (7), our computation according to the modified differential approximation will be valid only for a blackbody plate and hence A is strictly equal $\frac{3}{2}$ in this case.

† An approximate method, practically identical to Olfe's modification, has also been published by Glicksman [7].

For an application of the modified differential approximation to gray walls we refer to [7]. Contrary to this, our solution according to the unmodified differential approximation also accounts for a gray plate.

SOLUTION OF THE BOUNDARY VALUE PROBLEM

(a) *Application of the unmodified differential approximation*

In [3], p. 277, Cess points out the equivalence of the radiation-layer problem with the one-dimensional transient radiant heating problem. Although Arpaci [11] has given small and large time solutions for the transient problem (including the generalization for a non-gray gas and non-gray boundaries), the author could not find any explicit solution for the differential approximation formulation in the literature. Therefore it is given here for reference and comparison with the exponential-kernel solution of [3].

Applying the Laplace transformation with respect to x to equations (4) and (5) (with due consideration of (9)) and to the boundary conditions at infinity and on the plate (equation (8) with $\overline{\theta'_p}$ eliminated by means of the energy equation), we find after solving the ordinary differential problem

$$\overline{q'}(y;p) = \frac{4A}{3}\frac{1}{p}\frac{m}{m + ABu}\,e^{-my},$$

$$\overline{\theta'}(y;p) = \frac{16ABu/Bo}{p(p + 16Bu/Bo)}\frac{e^{-my}}{A + [3p/(p + 16Bu/Bo)]^{\frac{1}{2}}};$$

$m(p)$ is defined in the Nomenclature. If $y = 0$, explicit inversion is possible by means of the convolution theorem and tables of Laplace transforms (e.g. [12]). For the numerical evaluation of the results it is advantageous to use the identify (see [13], number 6.611–4)

$$\frac{8A\sqrt{3}}{3 - A^2}\int_0^\infty I_0(4t)\exp\left[-4\frac{3 + A^2}{3 - A^2}t\right]dt = 1.$$

Here $I_0(z)$ is the modified Bessel function of order zero. Then, the explicit results for the heat transfer and temperature on the plate in terms of the variable ξ can be written

$$q'(\xi,0) = \frac{4A}{3 - A^2}\left\{\frac{8A\sqrt{3}}{3 - A^2}\int_\xi^\infty e^{-4t}I_0(4t) \times \exp\left[-\frac{8A^2}{3 - A^2}(t - \xi)\right]dt - \frac{A}{\sqrt{3}}e^{-4\xi}I_0(4\xi)\right\},$$

$$\theta'(\xi,0) = 1 - \frac{8A\sqrt{3}}{3 - A^2}\int_\xi^\infty e^{-4t}I_0(4t) \times \exp\left[-\frac{8A^2}{3 - A^2}(t - \xi)\right]dt.$$

(b) *Application of Olfe's modified differential approximation*

Substituting (6) for q' in the energy equation and writing q'_g instead of q' in (5) again we apply the Laplace transformation with respect to x and upon combining these two equations into one for $\overline{q'_g}$ we obtain

$$\frac{d^2\overline{q'_g}}{dy^2} - m^2\,\overline{q'_g} = -\frac{16\,Bu/Bo}{p(p + 16\,Bu/Bo)}\frac{d^2q'_{ext}}{dy^2}. \qquad (10)$$

While the equation for $\overline{q'}$ in the unmodified case was homogeneous, a source term due to the wall radiation q'_{ext} appears in the above equation for $\overline{q'_g}$. Hence the general solution of (10) is

$$\overline{q'_g}(y;p) = C_1(p)\,e^{-my} + C_2(p)\,e^{my} + \overline{q'_g}_{part}.$$

A particular solution $\overline{q'_g}_{part}$ is obtained by the method of variation of parameters, i.e. after integrating by parts once,

$$\overline{q'_g}_{part} = \frac{16\,Bu^2/Bo}{mp(p + 16\,Bu/Bo)}\left\{-2\sinh my + m\left[e^{my}\int_0^y e^{-mt}E_2(Bu\cdot t)\,dt + e^{-my}\int_0^y e^{mt}E_2(Bu\cdot t)\,dt\right]\right\}.$$

The constants of integration C_1 and C_2 are determined by applying the Laplace transformed boundary conditions at infinity and on the plate (i.e. equation (8) with the wall radiation term omitted and the temperature eliminated by means of the energy equation). Thus

$$C_1 = \frac{16Bu}{3Bo}\frac{1}{p^2}\frac{[A\sqrt{(p + 16Bu/Bo)} - \sqrt{3p}]\log\{1 + \sqrt{[3p/(p + 16Bu/Bo)]}\} - 2A\sqrt{3p}}{A\sqrt{(p + 16Bu/Bo)} + \sqrt{3p}},$$

$$C_2 = \frac{16Bu}{3Bo}\frac{1}{p^2}\log\{1 + \sqrt{[3p/(p + 16Bu/Bo)]}\}.$$

Once $\overline{q'_g}$ is found, the energy equation immediately gives $\overline{\theta'}$. As before, we are mostly interested in finding the heat flux and the gas temperature on the plate. In this case the only singularity of the above functions in the complex plane is a branch line between $p = 0$ and $p = -16\,Bu/Bo$, so that the inversion contour can be deformed into a loop surrounding the branch cut similar to the one depicted in [11]. The integration along the large circle and the small circle about $p = -16\,Bu/Bo$ gives no contribution in the limit as the radius goes to infinity and zero respectively. Upon introducing the variable ξ again, the remaining integration in conjunction with (7) yields the final result

$$q'(\xi, 0) = \frac{2A}{3\pi}\int_0^1 \frac{e^{-8\xi t}\left\{\sqrt{\left(\frac{3t}{1-t}\right)}\log\sqrt{\left(1+\frac{3t}{1-t}\right)} - A\left[\operatorname{arctg}\sqrt{\left(\frac{3t}{1-t}\right)} - \sqrt{\left(\frac{3t}{1-t}\right)}\right]\right\}}{t^2\left(A^2+\frac{3t}{1-t}\right)}\,dt.$$

Similary, we find for the gas temperature on the plate

$$\theta'(\xi, 0) = \frac{1}{2}\left(1+\frac{3}{2A}\right) - \frac{1}{2\pi}\int_0^1 \frac{e^{-8\xi t}\left\{\sqrt{\left(\frac{3t}{1-t}\right)}\log\sqrt{\left(1+\frac{3t}{1-t}\right)} - A\left[\operatorname{arctg}\sqrt{\left(\frac{3t}{1-t}\right)} - \sqrt{\left(\frac{3t}{1-t}\right)}\right]\right\}}{t^2\left(A^2+\frac{3t}{1-t}\right)(1-t)}\,dt.$$

Figures 1 and 2 show the above results for the modified and unmodified differential approximation for a blackbody plate ($A = 3/2$) in comparison with the tabulated values of the exact solution [7] and the exponential kernel solution [3]. The latter is essentially equivalent to the unmodified differential approximation, the only difference being the choice of the parameters a, b in approximating $E_2(z)$ by $a\,e^{-bz}$. The coincidence of the exponential kernel result for the temperature with the exact result appears to be fortuitous. We note that the exponential kernel solution is asymptotically correct for small ξ while the differential approximation gives the asymptotically correct results for large ξ. As can be seen from Figs. 1 and 2 the solution according to Olfe's modification is a close approximation over the entire ξ range and gives the exact limits for $\xi = 0$ and $\xi \to \infty$ where it coincides asymptotically with the radiation slip solution (this can be checked easily by expanding for small p in the complex plane before inverting).

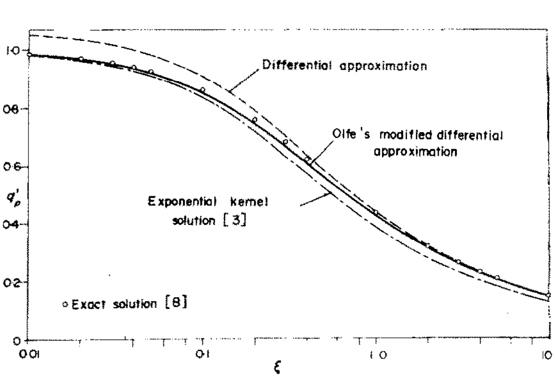

FIG. 1. Nondimensional linearized radiative heat flux $q'(\xi, 0)$ on a blackbody plate ($A = \frac{3}{2}$) as a function of ξ.

The good agreement adds further confidence to the claim that Olfe's modified differential approximation should yield not only qualitatively but also quantitatively quite accurate results for the two-dimensional radiation field of the hot-plate problem. Hence, such a solution, presently under study by the author, should be quite valuable in quantitatively establishing the range of validity for the one-dimensional model.

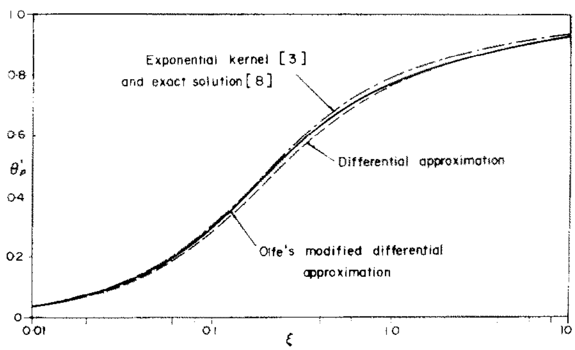

FIG. 2. Nondimensional gas temperature $\theta'(\xi, 0)$ on a blackbody plate ($A = \frac{3}{2}$) as a function of ξ.

ACKNOWLEDGEMENTS

The author is indebted to Dr. W. Schneider for suggesting the investigation of radiation–convection interaction effects and for many helpful discussions as well as critically reading the manuscript. Also Mr. L. Leopold's programming of the results on the CD 6400 is gratefully acknowledged.

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Int. J. Heat Mass Transfer. Vol. 15, pp. 2667–2670. Pergamon Press 1972. Printed in Great Britain

ON HEAT TRANSFER IN MHD CHANNEL FLOW

P. SCHROEDER*

Lehrstuhl und Institut fur Strömungsmechanik, Technische Universität, München, Germany

(*Received* 17 *April* 1972)

NOMENCLATURE

a, half width of the channel;
B, rheological parameter of the Prandtl–Eyring model;
c, specific heat;
C, rheological parameter of the Prandtl–Eyring model;
E, electric field strength;
F, incomplete elliptic integral of the first kind;
${}_3F_2$, generalized second order hypergeometric function (Clausen function);
H, magnetic field strength;
Ha, $\sqrt{(SRe)}$, Hartmann number;
j, electric current density;
k, thermal conductivity;
$k_{(1)}$, $\sqrt{\{1-[Ha_2/\tau_2^{*\prime}(0)]^2\}}$, $k_{(2)}=\sqrt{\{1-[\tau_2^{*\prime}(0)/Ha_2]^2\}}$, modulus of Jacobian elliptic functions and integrals;
K, $-\dfrac{E_z}{\mu_e H_y u_0}$, external loading parameter;
Kz, $\dfrac{aC}{u_0}$, characteristic parameter;
n, power law exponent;
K^*, $K-\dfrac{1}{S}\dfrac{\partial p^*}{\partial x^*}$, parameter containing pressure gradient, Stuart number and external loading parameter;

* Present address: Department of Aeronautics and Astronautics, Stanford University, Stanford, California 94305, U.S.A.